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The Method of Lines and the Nonlinear Klein–Gordon Equation*

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1. INTRODUCTION

The purpose of this note is to present a new constructive existence proof for the Cauchy problem for certain nonlinear perturbations of the classical wave equation,

$$u_{tt} - \Delta u = g(x, t, u) \tag{1.1}$$

on a bounded space region $\Omega \subset R^n$ for bounded and unbounded choices of the time t. It was shown by Strauss [10] that weak solutions u exist for all time if g is an arbitrary continuous function not depending explicitly on t such that

$$g(x, u)u \leqslant 0, \qquad \text{for all} \quad x \in \Omega, \text{ all } u \in R,$$

and such solutions are dissipative in the sense that the energy does not increase. His method involved approximation by Lipschitz perturbations, for which contraction and energy conservation principles are available, together with an L^1 convergence criterion permitting passage to the limit. The results of [10] are in fact more general and permit the addition of a time dependent forcing term of appropriate growth (cf. the Corollary of [10]) as well as unbounded Ω.

The physical importance of (1.1) has long been recognized in quantum field theory where the choice

$$g(x, u) = -\mu^2 u - u^3$$

corresponds to a meson potential (cf. Schiff [8]). For such equations, and more generally, for the Klein-Gordon equation where u^3 is replaced by $|u|^\rho u$, $\rho > 0$, the semi-discretization method of this paper gives weak solutions for all time. For more general choices of $g(x, t, u)$, viz.,

$$g(x, t, u) = f(x, t) - b\,|u|^\rho u - G(u), \tag{1.2}$$

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we shall construct solutions on a bounded space-time domain $D = \Omega \times (0, T)$ for $f \in L^2(D)$, for G a lipschitz continuous function and for $b > 0$.

A classical treatment of the Klein-Gordon equation for $\rho > 4$ was given by Jörgens [4] for the case $n = 3$, in which case a unique smooth solution exists for all time. Segal [9] proved the existence of weak solutions for all time; the general uniqueness question remains open for $n \geqslant 3$ (cf. Lions [5]). The existence results of this paper as summarized in Theorem 2.4 are already retrieved in [10]; however, our stable implicit divided difference scheme, wherein $\partial^2 u/\partial t^2$ is replaced by second order backward divided differences, leads to an elliptic boundary value problem at each time step, which has a monotone formulation if G is monotone. The solution of the algebraic discretizations of such elliptic equations is now well understood (cf. [7]). For an explicit numerical analysis of the Klein-Gordon equation, cf. Strauss and Vasquez [11].

Finally, we define, for the sequel, the following spaces. If B is an arbitrary Banach space, and $0 < T \leqslant \infty$,

$$\text{(i)}\quad L^\infty(0, T; B) = \{u(\cdot, t): u \in B \text{ for almost all } t \text{ and } \| u(\cdot, t)\|_B \in L^\infty(0, T)\} \tag{1.3}$$

with

$$\text{(ii)}\quad \| u \|_{L^\infty} = \operatorname{ess\,sup}_{0 \leqslant t \leqslant T} \| u(\cdot, t)\|_B ;$$

$$\text{(i)}\quad W^{1,\infty}(0, T; B) = \{u(\cdot, t): u, u_t \in L^\infty(0, T; B)\} \tag{1.4}$$

with

$$\text{(ii)}\quad \| u \|_{W^{1,\infty}} = \| u(\cdot, 0)\|_B + \operatorname{ess\,sup}_{0 \leqslant t \leqslant T} \| u_t(\cdot, t)\|_B .$$

$L^\infty(0, T; B)$ and $W^{1,\infty}(0, t; B)$ are known to be Banach spaces [5] with weak-* sequentially compact unit balls. For convenience, we take as norm in $H_0^1(\Omega)$ the completion of $C_0^\infty(\Omega)$, the equivalent [1] energy norm,

$$\| u \|^2_{H_0^1} = \int_\Omega | \nabla u |^2 \geqslant K \| u \|^2_{L_2} . \tag{1.5}$$

Finally, $\mathscr{D}'(\Omega)$ is the usual distribution space and $\langle \cdot, \cdot \rangle$ represents the usual duality pairing.

2. Main Results and the Semi-Discretization

Let F be a real-valued function on $[0, \infty)$ satisfying, for $p > 2$,

$$F'(s) = bs^{p/2-1}, \qquad b > 0, s \geqslant 0. \tag{2.1}$$

Note that the choices

$$F(s) = (2b/p)\, s^{p/2}, \qquad s \geqslant 0,$$

and $p = \rho + 2$ for $\rho > 0$ yield,

$$b \mid u \mid^{\rho} u = b \mid u \mid^{p-2} u = F'(u^2)u.$$

Let Ω be a bounded open subset of R^n and let the initial data functions u^0 and u^1 satisfy

$$u^0, u^1 \in H_0^1(\Omega) \cap L^p(\Omega) \tag{2.2}$$

and let f satisfy

$$f \in L^2(D) \tag{2.3}$$

where $D = \Omega \times (0, T)$. Finally, let G be a Lipschitz continuous function on R satisfying

$$G(0) = 0. \tag{2.4}$$

We are interested in the initial value, homogeneous boundary value problem defined by (1.1) and (1.2). The precise sense in which solutions are sought is given by the following definition of weak solution.

DEFINITION 2.1. A real measurable function u in D for $0 < T < \infty$ is said to be a weak solution of the initial-boundary value problem for (1.1, 1.2) if, for each $\varphi \in C_0^\infty(D)$,

(i) $\int_D [-u_t\varphi_t + \nabla u \cdot \nabla\varphi + (G(u) + F'(u^2)u - f)\varphi]\, dx\, dt = 0,$
and if u satisfies the regularity and initial conditions,

(ii) $u \in W^{1,\infty}(0, T; L^2(\Omega)) \cap L^\infty(0, T; H_0^1(\Omega) \cap L^p(\Omega)),$ (2.5)

(iii) $u_{tt} \in \mathscr{D}'(\Omega) \cap L^2(0, T; H^{-1}(\Omega) + L^q(\Omega)),$

(iv) $u(\cdot, 0) = u^0,\ u_t(\cdot, 0) = u^1.$

Here q satisfies $1/p + 1/q = 1$. The solution is said to hold for all time if u satisfies the global condition,

$$u \in W^{1,\infty}(0, \infty; L^2(\Omega)) \cap L^\infty(0, \infty; H_0^1(\Omega) \cap L^p(\Omega)), \tag{2.5ii'}$$

together with (2.5i, iii) for all $0 < T < \infty$ and (2.5iv).

Remark. The norm in $H_0^1(\Omega) \cap L^p(\Omega)$ is taken as the sum of the respective norms. $H^{-1}(\Omega) + L^q(\Omega)$ is the dual of $H_0^1(\Omega) \cap L^p(\Omega)$.

Suppose now that an integer $M \geqslant 2$ is specified and set $\Delta t = T/M$. For $m = 1,..., M - 1$, consider the sequence of quasi-linear Dirichlet problems obtained formally by applying to (2.5) an implicit time discretization formula defined by backward second order divided differences:

$$\begin{aligned}[u_{m+1} - 2u_m + u_{m-1}]/(\Delta t)^2 - \Delta u_{m+1} + G(u_{m+1}) \\ + F'(u_{m+1}^2)\, u_{m+1} = f_{m+1}\,. \end{aligned} \tag{2.6}$$

Here $u_{m+1} \in H_0^1(\Omega) \cap L^p(\Omega)$, for $m = 1, \ldots, M-1$, $u_0 = u^0$ and $u_1 = u_0 + \Delta t u^1$. f_{m+1} is defined by,

$$f_{m+1} = \frac{1}{\Delta t} \int_{m\Delta t}^{(m+1)\Delta t} f(\cdot, t)\, dt. \tag{2.7}$$

The general problem of the existence of solutions of (2.6) is addressed by

PROPOSITION 2.1. *Given $g \in H^{-1}(\Omega) + L^q(\Omega)$, the semilinear Dirichlet problem,*

$$\int_\Omega [\nabla u \cdot \nabla \varphi + (G(u) + F'(u^2)u + u/(\Delta t)^2)\varphi] = g(\varphi),$$
$$\textit{for all} \quad \varphi \in H_0^1(\Omega) \cap L^p(\Omega) \tag{2.8}$$

has a unique solution $u \in H_0^1(\Omega) \cap L^p(\Omega)$ for

$$\| G \|_{\mathrm{Lip}} \leqslant 1/(\Delta t)^2. \tag{2.9}$$

Here $\| G \|_{\mathrm{Lip}}$ is a minimal Lipschitz constant.

The proof of the proposition makes fundamental use of two lemmas.

LEMMA 2.2. *Let V be a reflexive, separable Banach space and let A be a mapping from V to its dual V' satisfying:*

(i) *A is bounded and hemicontinuous, i.e.,*

$$\lambda \to \langle A(v + \lambda w), z \rangle$$

is continuous from R into R for all $v, w, z \in V$;

(ii) *$v_k \rightharpoonup v$ weakly in V, $A(v_k) \rightharpoonup \chi$ weakly in V' and*

$$\limsup_{k\to\infty} \langle A(v_k), v_k \rangle \leqslant \langle \chi, v \rangle \Rightarrow \chi = A(v). \tag{2.10}$$

(iii) *A is coercive, i.e.,*

$$\frac{\langle A(v), v \rangle}{\| v \|_V} \to \infty \qquad \text{as} \quad \| v \|_V \to \infty.$$

Then A is a surjective mapping of V onto V'.

Proof. cf. [5, p. 173].[1]

LEMMA 2.3. *There exists a constant $c_1 > 0$ such that*

$$\left| \int_\Omega [F'(v_1^2)\, v_1 - F'(v_2^2)\, v_2] z \right| \leqslant c_1(\| v_1 \|_{L^p} + \| v_2 \|_{L^p})^{p-2}$$
$$\times \| v_1 - v_2 \|_{L^p} \| z \|_{L^p} \tag{2.11}$$

for $v_1, v_2, z \in L^p(\Omega)$.

[1] (2.10ii) is known as property (M). It is a generalization of monotonicity.

Proof. By (2.1) and the mean value theorem for derivatives,

$$|F'(s_1^2)s_1 - F'(s_2^2)s_2| \leqslant 2b(p-1)(|s_1| + |s_2|)^{p-2}|s_2 - s_1|. \quad (2.12)$$

Noting that,

$$\frac{1}{p/(p-2)} + \frac{1}{p} + \frac{1}{p} = 1,$$

and applying the generalized Hölder inequality to the integral,

$$\int_\Omega \{(|v_1| + |v_2|)^{p-2}|v_2 - v_1||z|\},$$

we obtain (2.11) with $c_1 = 2b(p-1)$.

Proof of Proposition 2.1. We identify the space V of Lemma 2.2 with $H_0^1(\Omega) \cap L^p(\Omega)$ and V' with $H^{-1}(\Omega) + L^q(\Omega)$. The mapping A is defined by,

$$A(v) = -\Delta v + G(v) + F'(v^2)v + v/(\Delta t)^2. \quad (2.13)$$

where $-\Delta$ is viewed as a continuous bijection of $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$. Lemma 2.2, together with the characterization,

$$\| w \|_{L^q} = \sup \left\{ \left| \int_\Omega wz \right| : \| z \|_{L^p} \leqslant 1 \right\},$$

shows that the mapping

$$v \to F'(v^2)v = B(v) \quad (2.14)$$

is continuous from $L^p(\Omega)$ into $L^q(\Omega)$, as is $G + I/(\Delta t)^2$. A is thus seen to be hemicontinuous and bounded and (2.10i) holds. The coerciveness relation (2.10iii) is deduced from (2.1), (2.9) and (2.11). Indeed,

$$\begin{aligned}\langle A(v), v\rangle &= \langle -\Delta v, v\rangle + (G(v) + v/(\Delta t)^2, v)_{L^2} + (F'(v^2), v^2)_{L^2} \\ &\geqslant \| v \|_{H_0^1}^2 + \left(\frac{1}{(\Delta t)^2} - \| G \|_{\mathrm{Lip}}\right) \| v \|_{L^2}^2 + b \| v \|_{L^p}^p \\ &\geqslant \min(1, b)[\| v \|_{H_0^1}^2 + \| v \|_{L^p}^p].\end{aligned}$$

If $\| v \|_{H_0^1 \cap L^p} \to \infty$ and $\| v \|_{L^p}$ remains bounded then (2.10iii) follows from the preceding inequality. Suppose, then, that $\| v \|_{L^p} \to \infty$ as $\| v \|_{H_0^1 \cap L^p} \to \infty$, where we assume, without loss of generality, that $\| v \|_{L^p} \geqslant 1$; in particular, $\| v \|_{L^p}^p \geqslant \| v \|_{L^p}^2$. We have,

$$\frac{\langle A(v), v \rangle}{\| v \|_{H_0^1 \cap L^p}} \geqslant \frac{\min(1, b)}{2} \frac{(\| v \|_{H_0^1} + \| v \|_{L^p})^2}{\| v \|_{H_0^1} + \| v \|_{L^p}}$$

so that (2.10iii) holds in this case also.

It remains to verify (2.10ii). In fact, the stronger result

$$v_k \rightharpoonup v, \quad A(v_k) \rightharpoonup \chi \Rightarrow A(v) = \chi \tag{2.15}$$

actually holds. Now the compact injection of $H_0^1(\Omega)$ into $L^2(\Omega)$ [1, p. 99] and the equivalence of continuity and weak continuity of $-\Delta$ [2, p. 422] reduces (2.15) to the statement

$$v_k \rightharpoonup v, \quad B(v_k) \rightharpoonup \psi \Rightarrow B(v) = \psi, \tag{2.16}$$

where the weak convergence of $B(v_k)$ is taken in $L^q(\Omega)$. The verification of (2.16) is standard (cf. [5, p. 12, Lemma 1.3]) so that (2.10ii) holds. The existence assertion of the proposition now follows from Lemma 2.2.

Noting that $F'(s^2)s$ is a monotone increasing function on R, we obtain, if v_1 and v_2 are solutions,

$$0 = \langle A(v_1) - A(v_2), v_1 - v_2 \rangle \geqslant \langle -\Delta(v_1 - v_2), v_1 - v_2 \rangle = \| v_1 - v_2 \|_{H_0^1}^2$$

so that $v_1 = v_2$. This completes the proof of the proposition.

We define, for each $M \geqslant 2$, the sequence $\{u^M\}$ by,

$$u^M(x, t) = u_m(x) + \frac{(u_{m+1}(x) - u_m(x))(t - m\,\Delta t)}{\Delta t} \tag{2.17}$$

if $m\Delta t \leqslant t < (m + 1)\,\Delta t$, $m = 0, 1,..., M - 1$. Note that u^M is simply the piecewise linear interpolate in t of the "points" $u_0 = u^0$, $u_1 = u_0 + \Delta t\, u^1$ and u_{m+1} defined by (2.6) for $1 \leqslant m \leqslant M - 1$ with $\Delta t = T/M$.

The major result of the paper may now be stated. We assume (2.1–2.4).

THEOREM 2.4. *The sequence $\{u^M\}$ is bounded in each of the spaces $W^{1,\infty}(0, T; L^2(\Omega))$ and $L^\infty(0, T; H_0^1(\Omega) \cap L^p(\Omega))$. In particular, there is a subsequence convergent in $L^2(D)$ to a solution u of (2.5). This solution may be continued[2] for all time if $f \equiv 0$ and $G(u)$ is linear with nonnegative slope, i.e., in the case of the nonlinear Klein-Gordon equation.*

[2] More precisely, a solution exists for all time in this case, with restriction a solution of the bounded time interval problem.

3. Stability of the Semi-Discretization

The major results of this section describe stability results which are directly interpreted as providing bounds for the sequence u^M in the appropriate function spaces.

THEOREM 3.1. *There exist positive constants C_1 and C_2 such that, for $M \geqslant M_0$ and $m = 1,..., M-1$, the estimates,*

$$\begin{aligned} &\text{(i)} \quad \| u_{m+1} - u_m \|_{L^2}/\Delta t \leqslant C_1, \\ &\text{(ii)} \quad \| u_{m+1} \|_{H_0^1 \cap L^p} \leqslant C_2, \end{aligned} \tag{3.1}$$

hold for $\Delta t = T/M$, $0 < T < \infty$. Here $M_0 = \max(2, (\| G \|_{\mathrm{Lip}})^{1/2} T)$.

Proof. The basic starting point is the weak form of (2.6). For $m = 0,..., M-1$, define $y_m = u_{m+1} - u_m$. We have from (2.6), for $1 \leqslant k \leqslant M-1$,

$$\begin{aligned} \sum_{m=1}^{k} \Big\{ \Big(\frac{1}{\Delta t}\Big)^2 (y_m - y_{m-1}, y_m)_{L^2} + (u_{m+1}, y_m)_{H_0^1} \\ + (G(u_{m+1}), y_m)_{L^2} + \langle F'(u_{m+1}^2)\, u_{m+1}, y_m \rangle \Big\} \\ = \sum_{m=1}^{k} (f_{m+1}, y_m)_{L^2}. \end{aligned} \tag{3.2}$$

Now observe that, by (2.1) and the Hölder inequality,

$$\begin{aligned} I_{m+1} &= \int_\Omega F'(u_{m+1}^2)\, u_{m+1}(u_{m+1} - u_m) \\ &\geqslant b \| u_{m+1} \|_{L^p}^p - b \| u_{m+1} \|_{L^p}^{p/q} \| u_m \|_L \end{aligned}$$

and an application of the inequality [6, p. 213],

$$a_1^{1/p} a_2^{1/q} \leqslant \frac{a_1}{p} + \frac{a_2}{q}, \qquad a_1 \geqslant 0,\ a_2 \geqslant 0,$$

with $a_1 = \| u_m \|_{L^p}^p$, $a_2 = \| u_{m+1} \|_{L^p}^p$ gives the inequality,

$$I_{m+1} \geqslant \frac{b}{p} \| u_{m+1} \|_{L^p}^p - \frac{b}{p} \| u_m \|_{L^p}^p. \tag{3.3}$$

Note here that $p - 2 > 0$ and $2 - q > 0$. Thus we obtain from (3.3),

$$I_m + I_{m+1} \geqslant \frac{b}{p} \| u_{m+1} \|_{L^p}^p - \frac{b}{p} \| u_{m-1} \|_{L^p}^p \tag{3.4}$$

for $m = 2,..., M-1$.

Now, the inequalities,

$$(y_m - y_{m-1}, y_m)_{L^2} \geqslant \tfrac{1}{2}\| y_m \|_{L^2}^2 - \tfrac{1}{2}\| y_{m-1} \|_{L^2}^2,$$

$$(u_{m+1}, y_m)_{H_0^1} \geqslant \tfrac{1}{2}\| u_{m+1} \|_{H_0^1}^2 - \tfrac{1}{2}\| u_m \|_{H_0^1}^2,$$

$$|(G(u_{m+1}), y_m)_{L^2}| \leqslant \tfrac{1}{2}\left(4 \| G \|_{\mathrm{Lip}}^2 \| u_{m+1} \|_{L^2}^2 + \tfrac{1}{4}\left(\frac{1}{\Delta t}\right)^2 \| y_m \|_{L^2}^2\right) \Delta t,$$

$$|(f_{m+1}, y_m)_{L^2}| \leqslant \tfrac{1}{2}\left(4 \| f_{m+1} \|_{L^2}^2 + \tfrac{1}{4}\left(\frac{1}{\Delta t}\right)^2 \| y_m \|_{L^2}^2\right) \Delta t,$$

together with (3.2) and (3.4), lead to, for $\max(1, 8 \| G \|_{\mathrm{Lip}}^2) \leqslant 1/\Delta t \cdot \min(1, K)$,

$$\begin{aligned}\left(\frac{1}{4}\right)\left(\frac{1}{\Delta t}\right)^2 \| y_k \|_{L^2}^2 + \tfrac{1}{4} \| u_{k+1} \|_{H_0^1}^2 + \frac{b}{p} \| u_{k+1} \|_{L^p}^p \\ \leqslant \frac{b}{p} \| u_1 \|_{L^p}^p + 2 \| f \|_{L^2(D)}^2 + \tfrac{1}{2}[\| u^1 \|_{L^2}^2 + \| u_1 \|_{H_0^1}^2] \\ + (\Delta t) \sum_{m=1}^{k-1} \{\tfrac{1}{4} \| y_m \|_{L^2}^2/(\Delta t)^2 + 2 \| G \|_{\mathrm{Lip}}^2 \| u_{m+1} \|_{H_0^1}^2\}.\end{aligned} \tag{3.6}$$

Defining $Y^M(t)$ by the left hand side of (3.6) for $k\Delta t \leqslant t < (k+1)\Delta t$ and $1 \leqslant k \leqslant M - 1$ and by 0 for $0 \leqslant t < \Delta t$, we see from (3.6) that,

$$Y^M(t) \leqslant K_1 + K_2 \int_0^{k\Delta t} Y^M(s)\, ds, \qquad k\,\Delta t \leqslant t < (k+1)\,\Delta t, \tag{3.7}$$

where K_1 is any nonnegative constant bounding the sum of the first four terms on the right hand side of (3.6); this is possible since $u_1 = u^0 + \Delta t u^1$. A standard application of the Gronwall inequality for step functions (cf. [3, Lemma 3.3]) yields, for all but a finite number of M,

$$Y^M(t) \leqslant K_1 \exp[K_2 k\, \Delta t], \qquad k\Delta t \leqslant t < (k+1)\,\Delta t, \quad 0 \leqslant k \leqslant M - 1. \tag{3.8}$$

(3.8) immediately implies (3.1) and the Theorem is proved.

As a corollary we have,

COROLLARY 3.2. *The sequence* $\{u^M\}$ *defined by* (2.17) *is bounded in each of the spaced* $W^{1,\infty}(0, T; L^2(\Omega))$ *and* $L^\infty(0, T; H_0^1(\Omega) \cap L^p(\Omega))$.

Another corollary, of the proof of Theorem 3.1, applies to the case of the Klein-Gordon equation for unbounded time, i.e., for $m = 1, 2,\ldots$ and $\Delta t = 1/M$ $(M = 1, 2,\ldots)$. Slight modifications in the proof in this case yield $K_2 = 0$ so that C_1 and C_2 depend only upon the initial data.

COROLLARY 3.3. *In the case of the nonlinear Klein-Gordon equation, i.e., $f \equiv 0$ and $G(u) = \mu^2 u$, the relations* (3.1) *hold for all time. Specifically, there exist positive constants C_1 and C_2 such that* (3.1) *holds for $\Delta t = 1/M(M = 1, 2,\ldots)$ and $m = 1, 2,\ldots$. In particular, the corresponding sequence $\{u^M\}$ defined for all time is bounded in each of the spaces $W^{1,\infty}(0, \infty; L^2(\Omega))$ and $L^\infty(0, \infty; H_0^1(\Omega) \cap L^p(\Omega))$.*

Consider the sequence of step functions $\{\bar{u}^M\}$ defined by,

$$\bar{u}^M(x, t) = u_m(x), \qquad m\Delta t \leqslant t < (m+1)\Delta t, \quad m = 0,\ldots, M-1. \tag{3.9}$$

Then we have the following proposition (cf. [3, Proposition 3.8]).

PROPOSITION 3.4. *Suppose $\{M_i\}$ is a sequence of integers for which $\{u^{M_i}\}$ and $\{\bar{u}^{M_i}\}$ are both weakly convergent in $L^2(0, T; H_0^1(\Omega) \cap L^p(\Omega))$ for $0 < T < \infty$. Then the limits of these sequences coincide.*

The usefulness of this proposition is seen in the quadrature scheme determined by the semi-discretization for the integrals defining the weak solution.

We quote one final result.

LEMMA 3.5. *A set bounded in $W^{1,\infty}(0, T; L^2(\Omega))$ and $L^\infty(0, T; H_0^1(\Omega))$ for $0 < T < \infty$ is relatively compact in $L^2(D)$. In particular, the sequence $\{u^M\}$ has a subsequence convergent in $L^2(D)$.*

Proof. The lemma is a consequence of [5, p. 58, Th. 5.1] since the injection $H_0^1(\Omega) \to L^2(\Omega)$ is compact [1, p. 99] for an arbitrary bounded set Ω.

4. EXISTENCE OF REGULAR SOLUTIONS

In this section we shall outline the proof of Theorem 2.4; full details are omitted (cf. the similar technique in [3]).

Proof of Theorem 2.4. Suppose $0 < T \leqslant \infty$. By Corollary 3.2 (3.3 if $T = \infty$), Proposition 3.4 and Lemma 3.5, there is a subsequence $\{M_i\}$ of $\{M\}$ and a function

$$u \in L^\infty(0, T; H_0^1(\Omega) \cap L^p(\Omega)) \cap W^{1,\infty}(0, T; L^2(\Omega))$$

satisfying, for all $0 < T_0 < T$ and $D_0 = \Omega \times (0, T_0)$,

$$\begin{aligned}
&\text{(i)} \quad u^{M_i} \rightharpoonup^* u \ (\text{in } L^\infty(0, T_0; H_0^1(\Omega) \cap L^p(\Omega)),\\
&\text{(ii)} \quad \bar{u}^{M_i} \rightharpoonup^* u \ (\text{in } L^\infty(0, T_0; H_0^1(\Omega) \cap L^p(\Omega)),\\
&\text{(iii)} \quad u^{M_i} \rightharpoonup^* u \ (\text{in } W^{1,\infty}(0, T_0; L^2(\Omega))),\\
&\text{(iv)} \quad u^{M_i} \to u \ (\text{in } L^2(D_0)),\\
&\text{(v)} \quad u^{M_i}(x, t) \to u(x, t) \text{ a.e. in } D_0.
\end{aligned} \tag{4.1}$$

By the boundedness of $v \to F'(v^2)v$ as a mapping from $L^p(D)$ to $L^q(D)$ we may also assume that there is a function $\chi \in L^q(D)$ such that

$$F'([u^{M_i}]^2)\, u^{M_i} \rightharpoonup \chi \text{ (in } L^q(D)). \tag{4.1vi}$$

From the L^1 convergence lemma of [10] we conclude $F'(u^2)u = \chi$.

We now assume that $T < \infty$ is an integral multiple M of Δt, $M \geqslant 2$. Let $\varphi \in C_0^\infty(D)$, and define

$$\varphi_m(x) = \frac{1}{\Delta t} \int_{m\Delta t}^{(m+1)\Delta t} \varphi(x, t)\, dt, \qquad m = 0,..., M-1. \tag{4.2}$$

Using the weak from of (2.6) as a starting point, with test function φ_m, summing on m and applying summation by parts we obtain,

$$\sum_{m=1}^{M-2} \left(y_m/\Delta t, \frac{\varphi_m - \varphi_{m+1}}{\Delta t} \right)_{L^2} \Delta t + (y_{M-1}/\Delta t, \varphi_{M-1})_{L^2}$$
$$- (y_0/\Delta t, \varphi_1)_{L^2} + \sum_{m=1}^{M-1} (u_{m+1}, \varphi_m)_{H_0^1} + \sum_{m=1}^{M-1} (H_{m+1}, \varphi_m)_{L^2} \Delta t$$
$$= \sum_{m=1}^{M-1} (f_{m+1}, \varphi_m)_{L^2} \Delta t. \tag{4.3}$$

Here,

$$y_m = u_{m+1} - u_m, \qquad H_{m+1} = G(u_{m+1}) + F'(u_{m+1}^3)u_{m+1}, \quad m = 1,..., m-1.$$

Rewriting (4.3), we have,

$$\int_{\Delta t}^{(M-1)\Delta t} (Y^M/\Delta t, \zeta^M)_{L^2} + (y_{M-1}/\Delta t, \varphi_{M-1})_{L^2} - (y_0/\Delta t, \varphi_1)_{L^2}$$
$$+ \int_{2\Delta t}^{T} (u^M \varphi_{\Delta t}^M)_{H_0^1} + \int_{2\Delta t}^{T} (H^M, \varphi_{\Delta t}^M)_{L^2} = \int_{2\Delta t}^{T} (f^M, \varphi_{\Delta t}^M)_{L^2} \tag{4.4}$$

where Y^M, H^M, f^M and φ^M are the step functions determined by y_m, H_m, f_m and φ_m and where ζ^M is the negative of the forward divided difference of φ^M and where $\varphi_{\Delta t}^M$ is the translate of φ^M by $-\Delta t$.

Since

(i) $\varphi^M \to \varphi$ in $L^\infty(0, T; H_0^1(\Omega) \cap L^p(\Omega))$,

and (4.5)

(ii) $\zeta^M \to -\dfrac{\partial\varphi}{\partial t}$ in $W^{1,\infty}(0, T; L^2(\Omega))$,

(2.5i) follows readily from (4.4) as $M \to \infty$ through the sequence M_i.

(2.5ii) is proved in [5, pp. 8–9] and the initial conditions follow from the Bochner integral representations of $u(\cdot, t) - u^{M_i}(\cdot, t)$ and $u_t(\cdot, t) - u_t^{M_i}(\cdot, t)$ by standard arguments. The proof is now complete.

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